Mirror Descent Algortihms for Risk Bidgeting Portfolios

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Paper overview

- **Risk Budgeting** is a popular portfolio optimization framework (e.g., mean-variance).
- The objective is to **distribute portfolio risk** (e.g., volatility) among its assets according to pre-defined **risk budgets**.
- In this paper, we propose **Mirror Descent algorithms** to compute Risk Budgeting portfolios efficiently.
- The method extends the classical framework to general risk measures such as volatility, Expected Shortfall, and deviation measures.
- A **tamed-gradient approach** ensures numerical stability and convergence, even in stochastic settings.
- The resulting algorithms are simple, fast, and theoretically grounded, providing the **first non-asymptotic convergence rate for stochastic Risk Budgeting**.

What is Risk Budgeting (RB) portfolio?

RB focuses on the contribution of each asset to the portfolio risk, rather than the portfolio risk itself.

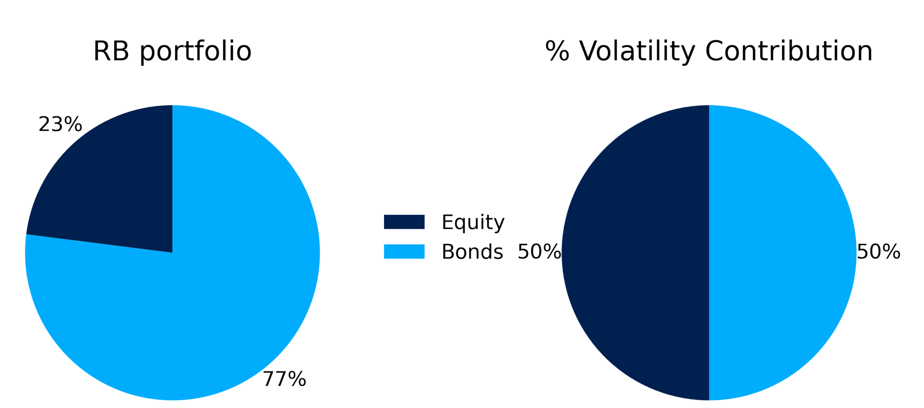
Mathematically, for a given vector of risk budgets b (whose elements are positive and add up to 1), a portfolio $(u)_{1\leq i\leq d}$ is the RB portfolio if the following condition is satisfied for every u_i where $r_\rho(u)$ denotes the portfolio risk:

$$u_i \partial_{u_i} r_\rho(u) = b_i \times r_\rho(u) \tag{1}$$

risk contribution of asset i = risk budget of asset i × portfolio risk

Example: 2-asset case

Let us define the portfolio risk $r_\rho(u)$ by volatility $\rho = \sigma(\theta)$. Given $b_1 = b_2 = 50\%$ and Σ of asset returns, we want to find the portfolio (on the left) which makes percentage volatility contributions (on the right) equal to risk budgets.



Which risk measures are suitable for RB?

A risk measure ρ is **RB-compatible** if it is

- **positive homogeneous**

$$\rho(\lambda Z) = \lambda \rho(Z), \quad \forall \lambda \geq 0$$

- **sub-additive**

$$\rho(Z_1 + Z_2) \leq \rho(Z_1) + \rho(Z_2)$$

Note: $r_\rho(u) = \rho(-\langle u, X\rangle)$

Classical formulation of the RB problem

Instead of dealing with the system of non-linear equations given by (1), we can find the RB portfolio by **minimizing** in $(\mathbb{R}_+^*)^d$:

$$\Gamma_g(y) = g\big(r_\rho(y)\big) - \sum_{i=1}^{d} b_i \log y_i \tag{2}$$

(g is convex and increasing) and normalizing the minimizer $u := \frac{y^*}{\sum_{i=1}^d y_i^*}$

Theoretical properties

- RB portfolio **exists** and is **unique**. (See cite complete proofs.)
- There exist a natural upper bound for the risk of RB portfolios.

$$r_\rho(u) \leq r_\rho(b)$$

When does the classical approach fail?

When $r_\rho(y)$ has **no closed form** (e.g. for Expected Shortfall), the gradient may **not be tractable**, making deterministic solvers unstable or slow. Thanks to (2), we can reformulated RB as a stochastic problem:

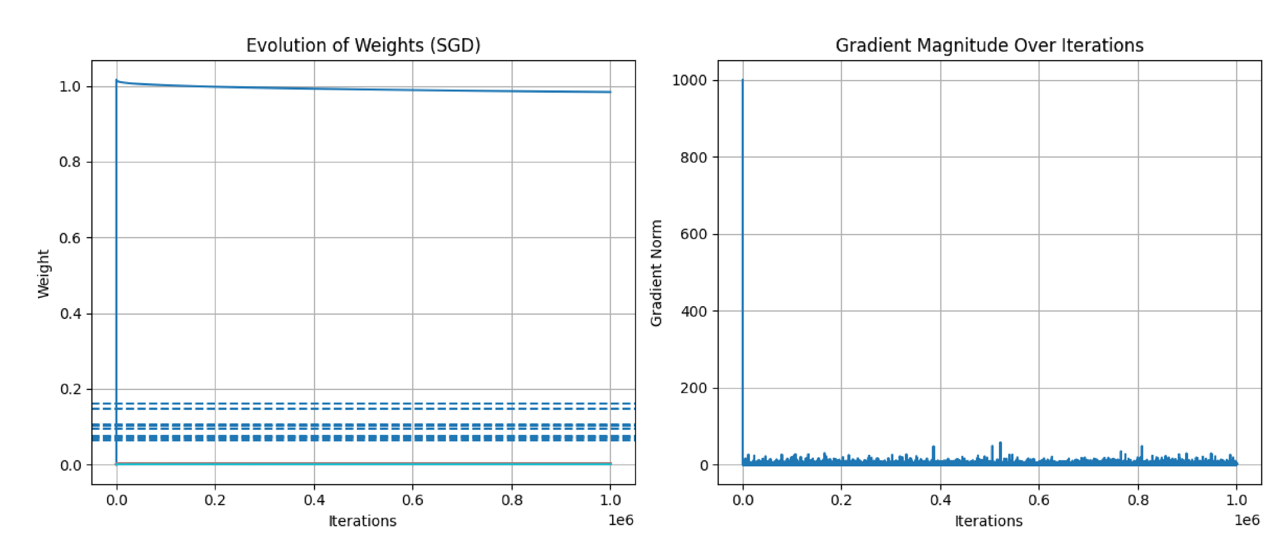
$$\min_{(y,\zeta)\in(\mathbb{R}_+^*)^d\times\mathbb{R}} \mathbb{E}\Big[\zeta + \frac{1}{1-\alpha}(-y'X - \zeta)_+ - \sum_{i=1}^{d} b_i \log y_i\Big]$$

Generally solved using **Gradient Descent** methods

However projected SGD algorithm (such as (3)) sometimes diverges because:

$$\nabla_{y_i}\Gamma_g(y) \text{ is } \textbf{unbounded} \text{ when } y_i \to 0.$$

Example:



From Instability to Stability: Mirror Descent with Taming

To fix this, we define a tamed gradient:

$$\kappa(y) = \underline{y} \wedge 1 \quad \text{where} \quad \underline{y} = \min_{1\leq i\leq d} y_i$$

so that

$$\nabla_t \Gamma_g(y) = \kappa(y)\nabla\Gamma_g(y)$$

which r**emains bounded while preserving the same minimizer**.

Moreover, instead of using Euclidean projections, Mirror Descent updates **iterates in the dual space using the Bregman divergence** (D_F) induced by negative entropy:

$$F(u) = \sum_i u_i \log u_i, \qquad D_F(u, u') = \sum_i u_i \log \frac{u_i}{u_i'}.$$

This guarantees all iterates remain positive — **no projection steps are needed!**

Stochastic Mirror Descent (SMD)

We rewrite the risk measure in variational form:

$$g(r_\rho(y)) = \min_{\xi\in\mathbb{R}} \mathbb{E}[L(\xi, -y^\top X)].$$

Hence, the Risk Budgeting problem becomes:

$$\min_{\xi,\, y\geq 0} \mathbb{E}\left[H(\xi, y, X) := L(\xi, -y^\top X) - \sum_i b_i \log y_i\right].$$

The **Stochastic Mirror Descent Algorithm (SMD)** algorithm updates, for step-size γ_{k+1} and sample X_{k+1}:

$$\begin{cases} \xi_{k+1} = \xi_k - \gamma_{k+1}\,\partial_\xi H(\xi_k, y_k, X_{k+1}), \\ y_{k+1} = P^m_{y_k}(\gamma_{k+1}\,\kappa(y_k)\,\nabla_y H(\xi_k, y_k, X_{k+1}))\,, \end{cases}$$

where $P^m_{y_k}(\cdot)$ is the mirror update ensuring positivity of the iterates.

Convergence of SMD

For step-sizes (γ_n) satisfying $\sum_n \gamma_n = \infty$ and $\sum_n \gamma_n^2 < \infty$, the stochastic iterates $Z_n = (\xi_n, u_n)$ **converge a. s. to the optimum** Z^*.

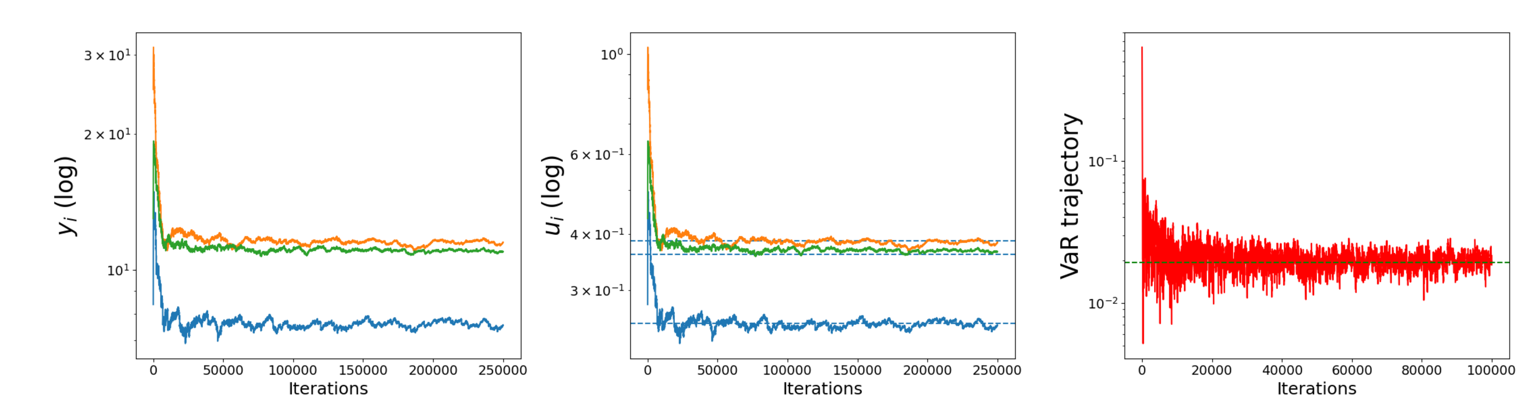
For the averaged sequence $\bar{Z}_n = \frac{\sum_{k=1}^n \gamma_k Z_{k-1}}{\sum_{k=1}^n \gamma_k}$, **decreases at rate**

$$\mathbb{E}\big[h(\bar{Z}_n) - h(Z^*)\big] = O(n^{-1/2}),$$

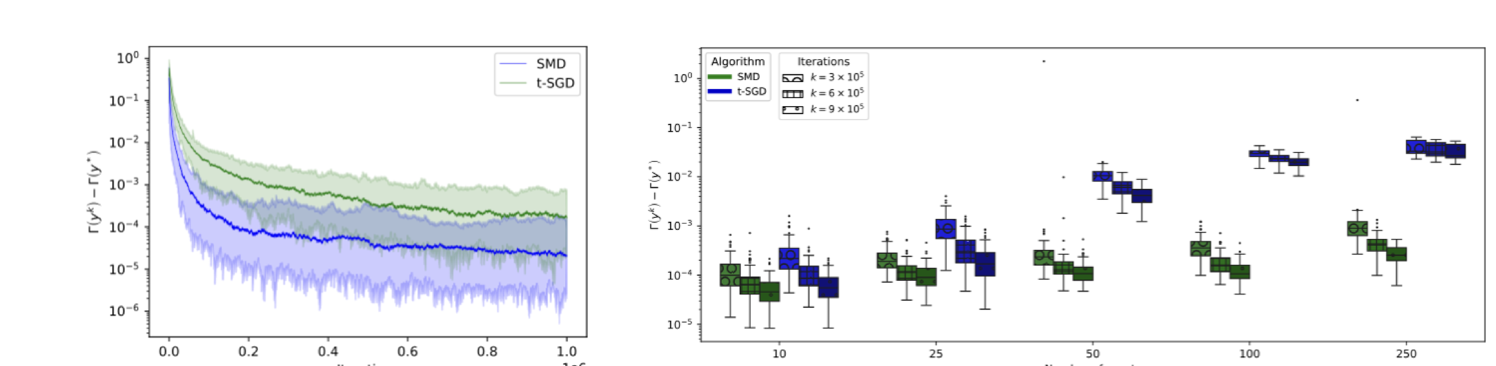
where $h(Z) = \mathbb{E}[H(Z, X)]$.

Numerical Illustration

Fast and stable portfolio optimization



Error evolution: SMD vs projected SGD



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